

PULLBACK OF PARABOLIC BUNDLES AND COVERS OF

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}$$

AJNEET DHILLON AND SHELDON JOYNER

ABSTRACT. We work over an algebraically closed ground field of characteristic zero. A G -cover of \mathbb{P}^1 ramified at three points allows one to assign to each finite dimensional representation V of G a vector bundle $\oplus \mathcal{O}(s_i)$ on \mathbb{P}^1 with parabolic structure at the ramification points. This produces a tensor functor from representation of G to vector bundles with parabolic structure that characterises the original cover. This work attempts to describe this tensor functor in terms of group theoretic data. More precisely, we construct a pullback functor on vector bundles with parabolic structure and describe the parabolic pullback of the previously described tensor functor.

1. INTRODUCTION

We work over an algebraically closed ground field k of characteristic zero. If G is a finite group then by [7] a G -torsor $f : X \rightarrow Y$ in the category of algebraic varieties can be thought of as a tensor functor $\text{Rep-}G \rightarrow \text{Vect}(Y)$. Concretely the associated tensor functor sends the representation V to the vector bundle $f_*(V \otimes \mathcal{O})^G$. When the cover ramifies, as was observed in [8], we need to consider tensor functors into the category of vector bundles with appropriate parabolic structure.

In the case where $Y = \mathbb{P}^1$ then we have $f_*(V \otimes \mathcal{O})^G = \oplus \mathcal{O}(s_i)$. The integers s_i are difficult to compute and one of our results is to find an upper bound on them when there is ramification at $0, 1$ and ∞ only. The bound 8.4, 8.6, improves the known bound in [3]. There is one case in which it is easy to compute the integers s_i , namely when the group G is cyclic. Our method is a kind of reduction to the cyclic case by removing ramification at 0 . More precisely, the endomorphism $z \mapsto z^n$ of \mathbb{P}^1 algebraically deloops loops around the origin. Pulling back a cover along this morphism removes ramification of order n at the origin. To make our method work we need to define a pullback morphism for parabolic bundles. As in [5] and [3] this entails use of an equivalence of categories due to Biswas, [2], between parabolic bundles of a certain kind, and vector bundles on an associated root stack. The pullback operation is difficult to reverse, that is given a morphism $f : X \rightarrow Y$ of smooth projective curves and a parabolic bundle \mathcal{F}_\bullet on X , to construct a parabolic bundle on Y that pulls back to \mathcal{F}_\bullet . In fact, the difficulty in reversing the parabolic pullback gives a new explanation for the fact that it is difficult to compute the s_i .

The interest in computing the s_i lies in the following. A finite quotient $q : F_2 \twoheadrightarrow G$ of the free group on two letters produces a cover $X_q \rightarrow \mathbb{P}^1$ ramified at three points. The absolute Galois group $G_{\mathbb{Q}}$ of \mathbb{Q} acts faithfully on such covers. However, given q , the Galois action is difficult to understand, and it is not known what finite quotient

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of $G_{\mathbb{Q}}$ acts, sending the cover to some other non-isomorphic cover. One way to try to understand this question is to give a more algebraic construction of the cover. The theory of tannakian categories allows one to do this. One should view the cover as a tensor functor into parabolic bundles and then understand the Galois action on such tensor functors. This work should be seen as a first step towards understanding these tensor functors. In this paper we understand their parabolic pullbacks. To understand the original functor amounts to faithfully flat descent for parabolic bundles. This will be a topic of future work.

In section two we recall some results of Nori on principal bundles and tensor functors. The third section recalls the notion of root stack introduced in [4]. Section four introduces parabolic bundles in our context. The definition here is equivalent to the one in [6]. We also recall from [11] the construction of tensor product and internal hom for parabolic bundles. Section five is devoted to proving the orbifold-parabolic correspondence in our context. This result is not new and goes back to [2]. The formulation here is based on the results of [3].

The new results begin in section six. We describe a construction on parabolic bundles that corresponds to pullback of orbifold bundles. In section seven we use some combinatorics to describe the case of cyclic covers. The final section gives an upper bound on the integers s_i described above, in the case of a G -cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The group G need not be abelian here.

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NOTATIONS AND CONVENTIONS

- (i) k an algebraically closed field of characteristic 0.
- (ii) X a connected smooth projective curve over k .
- (iii) For $x \in \mathbb{R}$ denote by $\lfloor x \rfloor$ the floor of x , i.e. the largest integer smaller than x .

2. SOME RESULTS OF NORI

In this section we recall some results from [7] and [8]. We begin by recalling the notion of a tannakian category. For a less terse formulation refer to [10] or [9].

Let L be a field. We denote by $\text{Vect}(L)$ the category of finite-dimensional L -vector spaces.

Definition 2.1. A *tannakian category* over L consists of a quadruple $(\mathbf{C}, \otimes, F, U)$ where

- T1. \mathbf{C} is a small, L -linear, abelian category.
- T2. $F : \mathbf{C} \rightarrow \text{Vect}(L)$ is an L -linear additive faithful exact functor called the fiber functor.
- T3. $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is an associative and commutative functor that is L -linear in each variable.
- T4. U is a unit for \otimes .

This data is subject to the following constraints:

- C1. F preserves \otimes .
- C2. F preserves the associativity and commutativity constraints.
- C3. $FU \xrightarrow{\sim} k$.
- C4. $\dim FV = 1$ if and only if there exists $V^{-1} \in \text{Objects}(\mathbf{C})$ such that $V \otimes V^{-1} \cong U$.

Remark 2.2. One uses [9, Proposition 1.20] to see that the category \mathbf{C} is necessarily rigid.

If G is an affine group scheme over k then the category $\text{Rep-}G$ of finite dimensional left representations of G is a tannakian category over k . In fact :

Theorem 2.3. *Any tannakian category over k is equivalent to $\text{Rep-}G$ for some affine group scheme G over k . Under this correspondence a homomorphism of affine group schemes corresponds to a tensor functor that commutes with fiber functor and preserves units.*

For a scheme X over k denote by $\text{Vect}(X)$ the category of algebraic vector bundles over X . The category $\text{Vect}(X)$ is a k -linear tensor category. The tensor product is associative and commutative and has a unit. Taking the fibre over a k -point gives it the structure of a tannakian category.

Definition 2.4. A rigid tensor G -functor on X is an R -linear exact \otimes -functor $F : \text{Rep-}G \rightarrow \text{Vect}(X)$ such that

- F1. F commutes with \otimes
- F2. F preserves the associativity and commutativity constraint
- F3. $\text{rk} FV = \dim V$
- F4. $F(V_{\text{triv}}) = \mathcal{O}_X$

We denote the category of such functors by $\text{Func}^{\otimes}(\text{Rep-}G, \text{Vect}(X))$. A morphism in this category is a natural transformation $\eta : F \rightarrow G$ such that the following diagram commutes :

$$\begin{array}{ccc} \otimes_{i \in I} F(X_i) & \xrightarrow{\sim} & F(\otimes_{i \in I} X_i) \\ \eta \downarrow & & \downarrow \eta \\ \otimes_{i \in I} G(X_i) & \xrightarrow{\sim} & G(\otimes_{i \in I} X_i). \end{array}$$

Such a natural transformation is necessarily an isomorphism, [9, Proposition 1.13].

Given $P \rightarrow X$, a G -torsor, we obtain a natural functor

$$F_P \in \text{Func}^{\otimes}(\text{Rep-}G, \text{Vect}(X))$$

given by $V \mapsto P \times_G V$.

We denote by $\text{Bun}_{G,X}$ the category of G -torsors over X . Notice that all the morphisms in this category are isomorphisms.

Theorem 2.5. *There is an equivalence of categories*

$$\text{Bun}_{G,X} \xrightarrow{\sim} \text{Func}^{\otimes}(\text{Rep-}G, \text{Vect}(X)).$$

Proof. See [7]. □

We will mostly be interested in the case where G is a finite group and $X = \mathbb{P} \setminus \{0, 1, \infty\}$. To make this setup more useful in this case we need a ramified version

of this theorem. Such a theorem already exists in [8], but we wish to restate things in terms of stacks. For now let us record the following corollary.

Corollary 2.6. *Let H be another finite group acting on X . Denote by $\mathrm{Bun}_{G,X}^H$ the category of G -torsors with an action of H that commutes with the action of G . Then we have an equivalence of categories*

$$\mathrm{Bun}_{G,X}^H \xrightarrow{\sim} \mathrm{Func}^\otimes(\mathrm{Rep}\text{-}G, \mathrm{Vect}_H(X)).$$

Here $\mathrm{Vect}_H(X)$ is the category of H -vector bundles on X .

Proof. Given a G -torsor $P \rightarrow X$ with a commuting H -action we obtain for each $h \in H$ a tensor functor

$$F_h : \mathrm{Rep}\text{-}G \rightarrow \mathrm{Vect}(X).$$

But as the pullbacks $P \times_{X,h} X$ are all isomorphic the functors above are all isomorphic by the theorem so we obtain a functor into $\mathrm{Vect}_H(X)$.

Conversely suppose that we have a tensor functor

$$F : \mathrm{Rep}\text{-}G \rightarrow \mathrm{Vect}_H(X).$$

Ignoring the H -action we obtain a torsor $P \rightarrow X$. But now the pullbacks $P \times_{X,h} X$ are all isomorphic as the original bundles were H -bundles. \square

3. ROOT STACKS

In this section, we recall some constructions from [4].

We shall implicitly make use of the following fact throughout this section : to give a morphism from a scheme S to the quotient stack $[\mathbb{A}^k/\mathbb{G}_m^k]$ is the same as giving a tuple $(\mathcal{L}_i, s_i)_{i=1}^k$ of line bundles \mathcal{L}_i on S and sections $s_i \in \Gamma(S, \mathcal{L}_i)$, see [4, Lemma 2.1.1].

Given a k -tuple $\vec{r} = (r_1, \dots, r_k)$ of positive integers there is a morphism of quotient stacks

$$\theta_{\vec{r}} : [\mathbb{A}^k/\mathbb{G}_m^k] \rightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$$

induced by the morphism

$$\begin{aligned} \mathbb{A}^k &\rightarrow \mathbb{A}^k \\ (x_1, \dots, x_k) &\mapsto (x_1^{r_1}, \dots, x_k^{r_k}). \end{aligned}$$

Definition 3.1. Let $\mathbb{D} = (D_1, \dots, D_k)$ be a k -tuple of effective Cartier divisors on a scheme S . This data defines a morphism $S \rightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$. Define the root stack $S_{\mathbb{D}, \vec{r}}$ to be

$$S_{\mathbb{D}, \vec{r}} = S \times_{[\mathbb{A}^k/\mathbb{G}_m^k], \theta_{\vec{r}}} [\mathbb{A}^k/\mathbb{G}_m^k].$$

Remark 3.2. Let $f : T \rightarrow S$ be a morphism. A lift of f to a T -point of $S_{\mathbb{D}, \vec{r}}$ is the same as giving

$$(M_1, \dots, M_k, t_1, \dots, t_k, \phi_1, \dots, \phi_k)$$

where M_i are line bundles on T , ϕ_i are isomorphisms $M_i^{r_i} \xrightarrow{\sim} f^*\mathcal{O}(D_i)$ and t_i are global sections of M_i such that

$$\phi_i(t_i^{r_i}) = s_{D_i},$$

where s_{D_i} denotes the tautological section of $\mathcal{O}(D_i)$ vanishing along D_i .

Proposition 3.3. *Let Y be a smooth projective curve with an action of a finite group G . Let $\psi : Y \rightarrow Y/G = X$ be the projection and assume that the action is generically free. Let the ramification divisor of ψ be $p_1 + \dots + p_k$ with ramification indices r_1, \dots, r_k . Set $\mathbb{D} = (p_1, \dots, p_k)$ and $\vec{r} = (r_1, \dots, r_k)$. Then*

$$[Y/G] \xrightarrow{\sim} X_{\mathbb{D}, \vec{r}}.$$

Proof. Let $\pi : X_{\mathbb{D}, \vec{r}} \rightarrow X$ be the canonical morphism. Write

$$\psi^*(p_i) = r_i D_i.$$

Then the D_i produce a G -equivariant morphism

$$\alpha : Y \rightarrow X_{\mathbb{D}, \vec{r}}.$$

Hence the question that we have an isomorphism is local.

We consider an open affine $\text{Spec} A \subset X$ with preimage $\text{Spec} B \subset Y$. We may assume $p_1 \in \text{Spec} A$ and $p_i \notin \text{Spec} A$ for $i > 1$. Let s_{p_1} be a parameter at p_1 . Then $\pi^{-1}(\text{Spec} A)$ is the quotient stack

$$[\text{Spec}(A[t]/(t^{r_1} - s_{p_1})) / \mu_{r_1}],$$

see [4, Example 2.4.1]. We have a diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\quad\quad\quad} & Y \\ \downarrow & & \downarrow \\ \text{Spec}(A[t]/(t^{r_1} - s_{p_1})) & \xrightarrow{\quad\quad\quad} & X \end{array}$$

where \tilde{Y} is the normalization of Y restricted to $\text{Spec}(A[t]/(t^{r_1} - s_{p_1}))$. By Abhyankar's lemma, it is a G -torsor and hence we obtain a morphism

$$\text{Spec}(A[t]/(t^{r_1} - s_{p_1})) \rightarrow [Y/G].$$

Due to the fact that the torsor \tilde{Y} has a μ_r -action we see that this morphism gives a morphism

$$\beta : [\text{Spec}(A[t]/(t^{r_1} - s_{p_1})) / \mu_{r_1}] \rightarrow [Y/G].$$

We need to show that $\alpha \cdot \beta$ and $\beta \cdot \alpha$ are automorphisms. But this is easily checked. \square

Consider a pair (\mathbb{D}, \vec{r}) with $\mathbb{D} = (n_1 p_1, \dots, n_k p_k)$ and $\vec{r} = (r_1, \dots, r_k)$. We define

$$(\mathbb{D}, \vec{r})_{red} = \left((p_1, \dots, p_k), \left(\frac{r_1}{d_1}, \dots, \frac{r_k}{d_k} \right) \right)$$

where $d_i = \gcd(n_i, r_i)$.

Proposition 3.4. *There is a morphism*

$$X_{(\mathbb{D}, \vec{r})_{red}} \rightarrow X_{(\mathbb{D}, \vec{r})}.$$

Proof. Consider a scheme $f : S \rightarrow X$. A lift of f to a point of $X_{(\mathbb{D}, \vec{r})_{red}}$ corresponds to a tuple

$$(M_1, \dots, M_k, t_1, \dots, t_k, \phi_1, \dots, \phi_k),$$

where M_i are line bundles, with global sections t_i and isomorphisms

$$\phi_i : M_i^{r_i/d_i} \xrightarrow{\sim} f^* \mathcal{O}_X(p_i) \quad \phi_i t_i^{r_i/d_i} = s_{p_i}.$$

Here s_{p_i} is a section vanishing at p_i .

Now by [4, Remark 2.2.2], the lifting of a morphism of stacks $X_{(\mathbb{D}, \vec{r})_{red}} \rightarrow X$ to $X_{(\mathbb{D}, \vec{r})}$ is similar to the lifting of a morphism of schemes in that it entails the same data as given in our Remark 3.2 above.

Observe that

$$M_i^{n_i/d_i} \quad t_i^{n_i/d_i} \quad \phi_i^{n_i}$$

give the data of a morphism to

$$X_{(\mathbb{D}, \vec{r})}.$$

□

Proposition 3.5. *We work in the situation of proposition 3.3. Suppose that*

$$[Y/G] = X_{(\mathbb{D}, \vec{r})}.$$

*Consider $f : Z \rightarrow X$ with Z a smooth projective curve. Denote by $\widetilde{f^*Y}$ the normalization of the fibered product*

$$Z \times_X Y.$$

Then

$$[\widetilde{f^*Y}/G] = Z_{(f^*\mathbb{D}, \vec{r})_{red}}.$$

Proof. By the proof of (3.3) this result will follow once we have computed the ramification indices of the morphism

$$\widetilde{f^*Y} \rightarrow Z.$$

Infinitesimally locally the morphism $Y \rightarrow X$ is of the form $y \mapsto y^n$ and the morphism $Z \rightarrow X$ is of the form $z \mapsto z^m$. The pullback is the high order cusp $y^n = z^m$. This has $d = \gcd(n, m)$ branches in its resolution and a local calculation gives the result. □

We shall need the following result later :

Proposition 3.6. *Every vector bundle on $X_{(\mathbb{D}, \vec{r})}$ is locally a direct sum of line bundles. Furthermore, when $X = \text{Spec}(R)$ with R local then $\text{Pic}(X_{p,r})$ is cyclic of order r and is generated by the canonical root line bundle.*

Proof. See [3, Proposition 3.12] and its proof. □

Notation 3.7. We will denote the canonical root line bundles on $X_{(\mathbb{D}, \vec{r})}$ by

$$\mathcal{N}_1, \dots, \mathcal{N}_k.$$

4. PARABOLIC BUNDLES

Let $D = n_1 p_1 + \dots + n_k p_k$ be an effective divisor on X with $p_i \neq p_j$ for $i \neq j$ and $n_i \geq 0$. We denote by \mathbb{D} the tuple $(n_1 p_1, n_2 p_2, \dots, n_k p_k)$. Fix a tuple of integers $\vec{r} = (r_1, \dots, r_k)$ with $r_i \geq 1$. The set

$$\frac{1}{r_1} \mathbb{Z} \times \dots \times \frac{1}{r_k} \mathbb{Z}$$

has a natural partial ordering with

$$\left(\frac{x_1}{r_1}, \dots, \frac{x_k}{r_k} \right) \leq \left(\frac{y_1}{r_1}, \dots, \frac{y_k}{r_k} \right)$$

if and only if

$$\frac{x_i}{r_i} \leq \frac{y_i}{r_i}$$

for all i . We shall often denote the poset

$$\frac{1}{r_1}\mathbb{Z} \times \dots \times \frac{1}{r_k}\mathbb{Z}$$

by

$$\frac{1}{\vec{r}}\mathbb{Z}.$$

If $\vec{\alpha} = (\alpha_1, \dots, \alpha_k) \in \frac{1}{\vec{r}}\mathbb{Z}$ then there is a natural shift functor $[\vec{\alpha}]$ on the category of functors

$$\left(\frac{1}{r_1}\mathbb{Z} \times \dots \times \frac{1}{r_k}\mathbb{Z}\right)^{op} \rightarrow \text{Vect}(X)$$

given by precomposition with the addition functor

$$+\vec{\alpha} : \frac{1}{\vec{r}}\mathbb{Z} \rightarrow \frac{1}{\vec{r}}\mathbb{Z}.$$

Definition 4.1. A parabolic bundle supported on \mathbb{D} with \vec{r} -divisible weights is a functor

$$\mathcal{F}_\bullet : \left(\frac{1}{r_1}\mathbb{Z} \times \dots \times \frac{1}{r_k}\mathbb{Z}\right)^{op} \rightarrow \text{Vect}(X)$$

with natural isomorphisms

$$j_{\mathcal{F}_\bullet, i} : \mathcal{F}_\bullet \otimes \mathcal{O}(-n_i p_i) \xrightarrow{\sim} \mathcal{F}_\bullet[0, \dots, 0, 1, 0, \dots, 0]$$

(with 1 in the i th position) making the following diagram commute

$$\begin{array}{ccc} \mathcal{F}_\bullet(-n_i p_i) & \longrightarrow & \mathcal{F}_\bullet[0, \dots, 0, 1, 0, \dots, 0] \\ & \searrow & \swarrow \\ & \mathcal{F}_\bullet & \end{array}$$

This data is required to satisfy the following axioms:

- (i) If $\alpha_i \leq \alpha'_i \leq \alpha_i + 1$ for all i then $\text{coker}(\mathcal{F}_{\vec{\alpha}'} \hookrightarrow \mathcal{F}_{\vec{\alpha}})$ is a locally free \mathcal{O}_D -module. Here $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\vec{\alpha}' = (\alpha'_1, \dots, \alpha'_k)$.
- (ii) For every $\vec{\alpha} = (\alpha_1, \dots, \alpha_k) \in \frac{1}{\vec{r}}\mathbb{Z}$ we have that $\mathcal{F}_{\vec{\alpha}}$ is the fibered product of $\mathcal{F}_{([\alpha_1], \dots, [\alpha_{i-1}], \alpha_i, [\alpha_{i+1}], \dots, [\alpha_k])}$ over $\mathcal{F}_{([\alpha_1], \dots, [\alpha_k])}$, i.e

$$\mathcal{F}_{\vec{\alpha}} = \bigtimes_{\mathcal{F}_{([\alpha_1], \dots, [\alpha_k])}} \mathcal{F}_{([\alpha_1], \dots, [\alpha_{i-1}], \alpha_i, [\alpha_{i+1}], \dots, [\alpha_k])}$$

When the context is clear, we write $j_{\mathcal{F}_\bullet, i} = j_i$. The morphisms making up the functor

$$\vec{\alpha} \leq \vec{\beta} \quad \mathcal{F}_{\vec{\beta}} \rightarrow \mathcal{F}_{\vec{\alpha}}$$

are necessarily injective so the second axiom merely asserts that

$$\mathcal{F}_{\vec{\alpha}} = \bigcap \mathcal{F}_{(0, \dots, 0, \alpha_i, 0, \dots, 0)},$$

when $\alpha_i > 0$ and the intersection is as submodules of

$$\mathcal{F}_{(0, 0, \dots, 0)}.$$

Remark 4.2. When the underlying divisor is reduced, this definition is equivalent to the original definition of Mehta and Seshadri in [6]. To spell things out, a Mehta-Seshadri parabolic bundle with \vec{r} -divisible weights and parabolic structure along \mathbb{D} consists of a vector bundle \mathcal{E} and for each p_i a filtration of

$$\mathcal{E}_{n_i p_i} := \mathcal{E}_{p_i} \otimes \mathcal{O}_{X, p_i} / \mathfrak{m}_{p_i}^{n_i}$$

given by

$$\mathcal{E}_{n_i p_i} = F_{1,i}(\mathcal{E}_{n_i p_i}) \supsetneq \dots \supsetneq F_{m_{p_i}, i}(\mathcal{E}_{n_i p_i}) \supsetneq F_{m_{p_i}+1, i}(\mathcal{E}_{n_i p_i}) = 0$$

and rational numbers $(\alpha_{i,j})_{1 \leq j \leq m_{p_i}}$ of the form l/r_i satisfying

$$0 \leq \alpha_{i,1} < \dots < \alpha_{i,m_{p_i}} < 1$$

subject to the condition that

$$F_{j,i}(\mathcal{E}_{n_i p_i}) / F_{j+1,i}(\mathcal{E}_{n_i p_i})$$

be locally free as modules over $\mathcal{O}_{X, p_i} / \mathfrak{m}_{p_i}^{n_i}$.

Let \mathcal{F}_\bullet be a parabolic bundle as defined in 4.1. The quotients

$$\mathcal{F}_{(0, \dots, 0, l/r_i, 0, \dots, 0)} / \mathcal{F}_{(0, \dots, 0, 1, 0, \dots, 0)}$$

for $0 \leq l/r_i < 1$ define a filtration

$$F_{1,i}(\mathcal{F}_\bullet) \supsetneq F_{2,i}(\mathcal{F}_\bullet) \supsetneq \dots \supsetneq F_{n_i, i}(\mathcal{F}_\bullet) \supsetneq 0$$

of $\mathcal{F}_{(0, \dots, 0)} / \mathcal{F}_{(0, \dots, 0, 1, 0, \dots, 0)} = \mathcal{F}_{(0, \dots, 0)} \otimes \mathcal{O}(-n_i p_i)$. We attach weights $\alpha_{i,j}$ to $F_{j,i}(\mathcal{F}_\bullet)$ by setting $\alpha_{i,j} = l/r_i$ where l is maximal such that

$$F_{j,i}(\mathcal{F}_\bullet) = \mathcal{F}_{(0, \dots, 0, l/r_i, 0, \dots, 0)} / \mathcal{F}_{(0, \dots, 0, 1, 0, \dots, 0)}.$$

The process is clearly reversible.

Definition 4.3. A morphism of parabolic bundles is a natural transformation

$$\phi : \mathcal{F}_\bullet \rightarrow \mathcal{F}'_\bullet$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_\bullet(-n_i p_i) & \xrightarrow{\sim} & \mathcal{F}_\bullet[0, \dots, 0, 1, 0, \dots, 0] \\ \downarrow & & \downarrow \\ \mathcal{F}'_\bullet(-n_i p_i) & \xrightarrow{\sim} & \mathcal{F}'_\bullet[0, \dots, 0, 1, 0, \dots, 0] \end{array}$$

Denote by $\text{Vect}_{\text{par}}(\mathbb{D}, \vec{r})$ the category of \vec{r} -divisible parabolic bundles with parabolic structure along \mathbb{D} . By modifying constructions and arguments given in [11], it is possible to endow this category with the structure of rigid tensor category. This entails defining a suitable tensor product and internal hom, which we describe now.

We have an addition bifunctor

$$+ : \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{\text{op}} \times \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{\text{op}} \rightarrow \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{\text{op}}$$

Definition 4.4. Let \mathcal{E}_\bullet , \mathcal{F}_\bullet and \mathcal{P}_\bullet be parabolic bundles. There is hence a functor

$$\mathcal{E}_\bullet \oplus \mathcal{F}_\bullet : \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\text{op}} \times \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\text{op}} \rightarrow \text{Vect}(X).$$

A *bilinear* morphism from \mathcal{E}_\bullet and \mathcal{F}_\bullet to \mathcal{P}_\bullet is a natural transformation

$$\eta : \mathcal{E}_\bullet \oplus \mathcal{F}_\bullet \rightarrow \mathcal{P}_\bullet \circ +$$

such that for every local section $f \in F_{\vec{\alpha}}$ (resp. $e \in E_{\vec{\alpha}}$) there is a parabolic morphism induced from η

$$\mathcal{E}_\bullet \rightarrow \mathcal{P}[\vec{\alpha}]_\bullet \quad (\text{resp. } \mathcal{F}_\bullet \rightarrow \mathcal{P}[\vec{\alpha}]_\bullet).$$

As above, let $\vec{\alpha}$ denote $(\alpha_1, \dots, \alpha_k)$ and similarly for $\vec{\beta}$ and $\vec{\gamma}$.

Definition 4.5. Given parabolic bundles \mathcal{E}_\bullet and \mathcal{F}_\bullet in $\text{Ob}(\text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}))$, define a functor

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_\bullet : \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\text{op}} \rightarrow \text{Vect} X$$

by setting

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\vec{\alpha}} := \left(\bigoplus_{\vec{\beta} + \vec{\gamma} = \vec{\alpha}} \mathcal{E}_{\vec{\beta}} \otimes_{\mathcal{O}_X} \mathcal{F}_{\vec{\gamma}} \right) / R_{\vec{\alpha}}$$

where $R_{\vec{\alpha}}$ is the \mathcal{O}_X submodule of the direct sum, which is locally generated by the sections:

$$[\mathcal{E}_\bullet(\vec{\beta} \rightarrow \vec{\beta}')]x \otimes y - x \otimes [\mathcal{F}_\bullet(\vec{\gamma}' \rightarrow \vec{\gamma})]y$$

for any $\vec{\beta} + \vec{\gamma} = \vec{\beta}' + \vec{\gamma}' = \vec{\alpha}$ where $x \in \mathcal{E}_{\vec{\beta}}$, $y \in \mathcal{F}_{\vec{\gamma}'}$ and $[\mathcal{E}_\bullet(\vec{\beta} \rightarrow \vec{\beta}')]x$ denotes the morphism in $\text{Vect}(X)$ which is the image of the morphism $\vec{\beta} \rightarrow \vec{\beta}'$ in $(\frac{1}{\vec{r}}\mathbb{Z})^{\text{op}}$ under the functor \mathcal{E}_\bullet (similarly for $[\mathcal{F}_\bullet(\vec{\gamma}' \rightarrow \vec{\gamma})]y$); and

$$x - j_i^{\vec{\beta}, \vec{\gamma}} x$$

for $i = 1, \dots, k$, where $j_i^{\vec{\beta}, \vec{\gamma}}$ denotes the morphism

$$(1 \otimes j_{\mathcal{F}_\bullet, i}(\vec{\gamma})) \circ (j_{\mathcal{E}_\bullet, i}(\vec{\beta} - (0, \dots, 0, 1, 0, \dots, 0))^{-1} \otimes 1)$$

mapping

$$\begin{aligned} \mathcal{E}_{\vec{\beta}} \otimes \mathcal{F}_{\vec{\gamma}} &\rightarrow \mathcal{E}_{(\beta_1, \dots, \beta_{i-1}, \beta_i-1, \beta_{i+1}, \dots, \beta_k)} \otimes \mathcal{O}(-n_i p_i) \otimes \mathcal{F}_{\vec{\gamma}} \\ &\rightarrow \mathcal{E}_{(\beta_1, \dots, \beta_{i-1}, \beta_i-1, \beta_{i+1}, \dots, \beta_k)} \otimes \mathcal{F}_{(\gamma_1, \dots, \gamma_{i-1}, \gamma_i+1, \gamma_{i+1}, \dots, \gamma_k)}. \end{aligned}$$

Also define the morphism $\psi_{(\mathcal{E} \otimes \mathcal{F})_\bullet}^{\vec{\alpha}, \vec{\alpha}'} := (\mathcal{E} \otimes \mathcal{F})_\bullet(\vec{\alpha} \rightarrow \vec{\alpha}')$ from $(\mathcal{E} \otimes \mathcal{F})_{\vec{\alpha}}$ to $(\mathcal{E} \otimes \mathcal{F})_{\vec{\alpha}'}$ in $\text{Vect}(X)$ by specifying for local sections $x \in \mathcal{E}_{\vec{\beta}}$ and $y \in \mathcal{F}_{\vec{\gamma}}$ with $\vec{\beta} + \vec{\gamma} = \vec{\alpha}$, that

$$\begin{aligned} \psi_{(\mathcal{E} \otimes \mathcal{F})_\bullet}^{\vec{\alpha}, \vec{\alpha}'}(x \otimes y \mod R_{\vec{\alpha}}) &= ([\mathcal{E}_\bullet(\vec{\beta} \rightarrow \vec{\alpha}' - \vec{\gamma})]x) \otimes y \mod R_{\vec{\alpha}'} \\ &= x \otimes ([\mathcal{F}_\bullet(\vec{\gamma} \rightarrow \vec{\alpha}' - \vec{\beta})]y) \mod R_{\vec{\alpha}'}. \end{aligned}$$

Now for each i , it is possible to define the isomorphism j_i associated to the functor $(\mathcal{E} \otimes \mathcal{F})_\bullet$ as follows: Consider for $i = 1, \dots, k$,

$$J_{\vec{\alpha}}^i := \bigoplus_{\vec{\gamma}} (1 \otimes j_{\mathcal{F}_\bullet, i}(\vec{\gamma}))$$

mapping

$$\bigoplus_{\vec{\gamma}} \mathcal{E}_{(\vec{\alpha} - \vec{\gamma})} \otimes \mathcal{F}_{\vec{\gamma}} \otimes \mathcal{O}(-n_i p_i) \rightarrow \bigoplus_{\vec{\gamma}} \mathcal{E}_{(\vec{\alpha} - \vec{\gamma})} \otimes \mathcal{F}_{(\gamma_1, \dots, \gamma_{i-1}, \gamma_i+1, \gamma_{i+1}, \dots, \gamma_k)}.$$

Then $J_{\vec{\alpha}}^i(R_{\vec{\alpha}} \otimes \mathcal{O}(-n_i p_i)) = R_{(\alpha_1, \dots, \alpha_i+1, \dots, \alpha_k)}$. Hence J_{\bullet}^i descends to the quotient and we denote this morphism $j_{(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}, i}$.

Lemma 4.6. *With this data, $(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}$ is a parabolic bundle with a bilinear morphism*

$$\mathcal{E}_{\bullet} \oplus \mathcal{F}_{\bullet} \rightarrow (\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet} \circ +$$

that is universal for all bilinear morphisms.

Proof. It is easy to check that $((\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}, j_{(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}, i}) \in \text{Ob}(\text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}))$.

To see the universal property, notice as in [11] that the canonical maps

$$f_{\vec{\alpha}, \vec{\beta}} : \mathcal{E}_{\vec{\alpha}} \otimes_{\mathcal{O}_X} \mathcal{F}_{\vec{\beta}} \rightarrow (\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\vec{\alpha} + \vec{\beta}}$$

determine a canonical bilinear morphism

$$f_{\bullet, \bullet} : \mathcal{E}_{\bullet} \oplus \mathcal{F}_{\bullet} \rightarrow (\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet} \circ +$$

of \mathcal{E}_{\bullet} and \mathcal{F}_{\bullet} to $(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}$ via morphisms $f_{\bullet, \vec{\beta}} : \mathcal{E}_{\bullet} \rightarrow (\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\vec{\beta}}$ and $f_{\vec{\alpha}, \bullet} : \mathcal{F}_{\bullet} \rightarrow (\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\vec{\alpha}}$ defined respectively for each fixed local section $b \in \mathcal{F}_{\vec{\beta}}$ and $a \in \mathcal{E}_{\vec{\alpha}}$. Because the latter morphisms are canonical embeddings, it follows that any bilinear morphism of \mathcal{E}_{\bullet} and \mathcal{F}_{\bullet} to some parabolic bundle \mathcal{P}_{\bullet} factors uniquely through $(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet} \circ +$. \square

Definition 4.7. Given parabolic bundles \mathcal{E}_{\bullet} and \mathcal{F}_{\bullet} in $\text{Ob}(\text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}))$, define a functor

$$\mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}_{\bullet})_{\bullet} : \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{op} \rightarrow \text{Vect}(X)$$

by setting

$$\mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}_{\bullet})_{\vec{\alpha}} := \mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}[\vec{\alpha}]_{\bullet}),$$

the (vector bundle of) natural transformations from the functor \mathcal{E}_{\bullet} to the shifted functor $\mathcal{F}[\vec{\alpha}]_{\bullet}$. The morphism $\vec{\alpha} \rightarrow \vec{\beta}$ in $\left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{op}$ induces a natural transformation of $\mathcal{F}[\vec{\alpha}]_{\bullet}$ to $\mathcal{F}[\vec{\beta}]_{\bullet}$ (i.e. the shift $[\vec{\beta} - \vec{\alpha}]$), thereby inducing a natural transformation

$$\mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}_{\bullet})_{\vec{\alpha}} \rightarrow \mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}_{\bullet})_{\vec{\beta}}$$

which we regard as the image of $\vec{\alpha} \rightarrow \vec{\beta}$ under the functor $\mathcal{H}om(\mathcal{E}_{\bullet}, \mathcal{F}_{\bullet})_{\bullet}$.

Lemma 4.8. *For a given \mathbb{D} and \vec{r} , $\text{Vect}_{\text{par}}(\mathbb{D}, \vec{r})$ with the tensor product and internal hom defined above in 4.5 and 4.7 respectively, is a rigid tensor category.*

Proof. This follows from the same arguments used to prove Lemmas 3.5 and 3.6 (equation (3.2)) in [11], modified to accord with our definitions. \square

An alternative description of the tensor product was given in [1]. This is useful for computations, so for later use, we formulate it here. The definition hinges on the embedding $\tau : X \setminus D \rightarrow X$:

Definition 4.9. The BBN tensor of the parabolic bundles \mathcal{E}_{\bullet} and \mathcal{F}_{\bullet} is the functor

$$(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})_{\bullet}^{BBN} : \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{op} \rightarrow \text{Vect}(X)$$

sending $\vec{\alpha}$ to the subsheaf of $\tau_* \tau^*(\mathcal{E}_{\bullet} \otimes \mathcal{F}_{\bullet})$ generated by (the canonical images of) $\mathcal{E}_{\vec{\beta}} \otimes \mathcal{F}_{\vec{\gamma}}$ for all $\vec{\beta} + \vec{\gamma} = \vec{\alpha}$.

Since \mathcal{E}_\bullet and \mathcal{F}_\bullet are parabolic, the requisite axioms are automatically satisfied. To show that the BBN tensor gives a parabolic bundle, it remains to exhibit the isomorphisms j_i . Instead, we prove

Lemma 4.10. *For any $\tilde{\alpha} \in (\frac{1}{r}\mathbb{Z})^{op}$, and any parabolic bundles \mathcal{E}_\bullet and \mathcal{F}_\bullet ,*

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\tilde{\alpha}} \simeq (\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\tilde{\alpha}}^{BBN}.$$

Proof. Any bundle $\mathcal{E}_{\tilde{\beta}} \otimes \mathcal{F}_{\tilde{\gamma}}$ with $\tilde{\beta} + \tilde{\gamma} = \tilde{\alpha}$ maps into $\tau_* \tau^*(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)$, producing a mapping

$$\phi : \oplus_{\tilde{\beta} + \tilde{\gamma} = \tilde{\alpha}} \mathcal{E}_{\tilde{\beta}} \otimes \mathcal{F}_{\tilde{\gamma}} \rightarrow (\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\tilde{\alpha}}^{BBN}$$

which by construction is a surjection. It remains to show that $R_{\tilde{\alpha}} = \ker \phi$. Since these are sheaves, the question is local. It is then immediate from the definition of $R_{\tilde{\alpha}}$ in terms of local sections, that this sheaf is a subsheaf of the kernel. An induction argument shows the reverse inclusion: Let m denote the number of non-zero entries in a given element of the direct sum. Also, let $(x_{\beta\gamma})_{\beta\gamma}$ denote an element of the direct sum, where $x_{\beta\gamma}$ is a local section of $\mathcal{E}_{\tilde{\beta}} \otimes \mathcal{F}_{\tilde{\gamma}}$. Elements of the kernel for which $m = 2$ are in $R_{\tilde{\alpha}}$: If $(x_{\beta\gamma})_{\beta\gamma}$ is such an element, then denote the non-zero entries by x_{st} and x_{uv} . Here suppose firstly that $x_{st} = x_s \otimes x_t$ and $x_{uv} = x_u \otimes x_v$ - i.e. each is a pure tensor of local sections. Then the image under ϕ is $\phi(x_{st}) + \phi(x_{uv}) = 0$. Abusing notation, this means that $x_s \otimes x_t = -x_u \otimes x_v$, which necessarily admits an expression as $\mathcal{E}[u \rightarrow s](-x_u) \otimes x_t = (-x_u) \otimes \mathcal{F}[t \rightarrow v]x_t$ so that $(x_{\beta\gamma}) \in R_{\tilde{\alpha}}$. More generally, if the non-zero terms are not pure tensors, by choosing bases for the local sections, which give canonical bases for the tensor products, it is possible to carry out a similar argument. Now if it is known that elements of the kernel for which $m \leq n-1$ all lie in $R_{\tilde{\alpha}}$, the same is true for those with $m = n$. To show this, we remark that because of axiom (ii) of Definition 4.1, it suffices to consider $\tilde{\alpha}$ of the form of $(0, \dots, 0, a, 0, \dots, 0)$ for some a . Without loss of generality, we may suppose that $k = 2$ - i.e. the tuples $(a, 0)$ and $(0, b)$ need only be considered. Then for pure tensors as before, we obtain $x_{s_1} \otimes x_{t_1} + \dots + x_{s_{n-1}} \otimes x_{t_{n-1}} = -x_{s_n} \otimes x_{t_n}$ with x_{s_j} (resp. x_{t_j}) a local section of \mathcal{E}_{s_j} (resp. \mathcal{F}_{t_j}). But by adding suitable elements of $R_{\tilde{\alpha}}$ to each term, when $\tilde{\alpha} = (a, 0)$, we may assume that the $s_j = (s'_j, 0)$ and the $t_j = (t'_j, 0)$. We may take $s'_1 < \dots < s'_n$, so that $t'_n < \dots < t'_1$. But then $\mathcal{E}_{s_1} \supset \dots \supset \mathcal{E}_{s_n}$ while $\mathcal{F}_{t_1} \subset \dots \subset \mathcal{F}_{t_n}$. Consequently $x_{s_n} \otimes x_{t_n} \in \mathcal{E}_{s_1} \otimes \mathcal{F}_{t_{n-1}}$, so that $x_{s_n} \otimes x_{t_n} - \mathcal{E}[s_n \rightarrow s_{n-1}]x_{s_n} \otimes x_{t_n} \in R_{\tilde{\alpha}}$, and may be added to the right side to reduce to the case that $m = n-1$. The general case may be handled using local bases as before. \square

We define a parabolic bundle $\mathcal{O}_{X_\bullet} : (\frac{1}{r}\mathbb{Z})^{op} \rightarrow \text{Vect}(X)$ by setting

$$\begin{aligned} \mathcal{O}_{X(0, \dots, 0)} &= \mathcal{O}_X \\ \mathcal{O}_{X(0, \dots, 0, t, 0, \dots, 0)} &= \mathcal{O}_X(-np_i) \quad \text{for } t \in (0, 1]. \end{aligned}$$

It is easily seen that this bundle is a unit for the tensor product.

5. THE PARABOLIC - ORBIFOLD CORRESPONDENCE

Recall that $\mathcal{N}_1, \dots, \mathcal{N}_k$ denote the canonical line bundles on $X_{\mathbb{D}, \vec{r}}$ that are roots of $\mathcal{O}(n_i p_i)$. Following [2] and [3] we then define a functor

$$\mathbf{F}_{\mathbb{D}, \vec{r}} : \text{Vect}(X_{\mathbb{D}, \vec{r}}) \rightarrow \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r})$$

$$\mathcal{F} \mapsto \left[\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k} \right) \mapsto \pi_* (\mathcal{N}_1^{-l_1} \otimes \dots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}) \right].$$

Remark 5.1. This functor is in fact a tensor functor where the tensor product in the category of parabolic bundles is defined as in the last section. In order to prove this it is useful to use the description of the tensor product in [1]. Given two vector bundles \mathcal{F}_1 and \mathcal{F}_2 we need to show that the two parabolic bundles $\mathbf{F}(\mathcal{F}_1 \otimes \mathcal{F}_2)$ and $\mathbf{F}(\mathcal{F}_1) \otimes \mathbf{F}(\mathcal{F}_2)$ are isomorphic. Away from the support of \mathbb{D} the stack $X_{\mathbb{D}, \vec{r}}$ is isomorphic to the curve X . Hence both of these bundles are subbundles of $\tau_* \tau^* (\mathbf{F}(\mathcal{F}_1) \otimes \mathbf{F}(\mathcal{F}_2))$. We need to show that they are the same subbundle. This question is local so we reduce to the case of one parabolic point and $\mathcal{F}_i = \mathcal{N}^{a_i}$. This is now easily checked.

The main result of this section is:

Theorem 5.2. *The functor $\mathbf{F}_{\mathbb{D}, \vec{r}}$ is an equivalence of categories.*

The proof given below is entirely analogous with the proof given in [3].

We have a canonical isomorphism

$$\pi^* \mathcal{O}^\alpha(n_i p_i) \rightarrow \mathcal{N}_i^{\alpha r_i}$$

and a section

$$s \in \Gamma(X_{\mathbb{D}, \vec{r}}, \mathcal{N}_i).$$

This produces by adjointness a canonical morphism

$$\mathcal{O}(n_i p_i)^{\lfloor l/r_i \rfloor} \rightarrow \pi_* (\mathcal{N}_i^l).$$

Proposition 5.3. *The above morphism is an isomorphism.*

Proof. See [3, 3.11]. □

To proceed we need to recall the notion of a *universal wedge* in category theory. Let \mathbf{B} and \mathbf{C} be categories and consider a functor $F : \mathbf{B}^{\text{op}} \times \mathbf{B} \rightarrow \mathbf{C}$. A *wedge* of F is an object x of \mathbf{C} and a collection of morphisms $a_i : F(i, i) \rightarrow x$ which are *dinatural*, that is for every morphism $f : i \rightarrow j$ in \mathbf{B} the following diagram commutes

$$\begin{array}{ccccc}
 & & F(i, i) & & \\
 & \nearrow^{F(f^{\text{op}}, 1)} & & \searrow_{a_i} & \\
 F(j, i) & & & & x \\
 & \searrow_{F(1, f)} & & \nearrow_{a_j} & \\
 & & F(j, j) & &
 \end{array}$$

A smallest such wedge is called a *universal wedge*. If it exists we will denote it by $\int^I F(I, I)$.

Proposition 5.4. *Let $\mathcal{F}_\bullet \in \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r})$. The universal wedge*

$$\int^{\frac{1}{\vec{r}} \mathbb{Z}} \mathcal{N}_1^{l_1} \otimes \dots \otimes \mathcal{N}_k^{l_k} \otimes \pi^* \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)}$$

exists in $\text{Vect}(X_{(\mathbb{D}, \vec{r})})$.

Proof. The question is local as wedges are colimits. The proof in the local case is already in [3]. \square

We denote the functor arising from 5.4 by $\mathbf{G}_{\mathbb{D}, \vec{r}}$.

Proposition 5.5. *Let $\mathcal{F} \in \text{Vect}(X_{\mathbb{D}, \vec{r}})$. The natural map*

$$\mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \pi^* \pi_* (\mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}) \rightarrow \mathcal{F}$$

is dinatural in (l_1, \dots, l_k) .

Proof. The morphism in question comes by tensoring the counit of adjunction

$$\pi^* \pi_* (\mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}) \rightarrow \mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}.$$

It is relatively straightforward to show that the resulting morphism is dinatural. The details are spelled out in [3, Lemma 3.18]. \square

Corollary 5.6.

$$\mathbf{G}_{\mathbb{D}, \vec{r}} \circ \mathbf{F}_{\mathbb{D}, \vec{r}} \simeq 1.$$

Proof. By the proposition, there exists a natural transformation

$$\mathbf{G}_{\mathbb{D}, \vec{r}} \circ \mathbf{F}_{\mathbb{D}, \vec{r}} \rightarrow 1.$$

To show that it is an isomorphism we may argue locally. This argument can be found in [3, page 18]. \square

Finally we need to show that

$$\mathbf{F}_{\mathbb{D}, \vec{r}} \circ \mathbf{G}_{\mathbb{D}, \vec{r}} \simeq 1.$$

We have

$$\begin{aligned} & \pi_* \left(\mathcal{N}_1^{-m_1} \otimes \cdots \otimes \mathcal{N}_k^{-m_k} \otimes \int \mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \pi^* \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)} \right) \\ & \simeq \pi_* \left(\int \mathcal{N}_1^{l_1 - m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k - m_k} \otimes \pi^* \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)} \right) \\ & \simeq \int \pi_* \left(\mathcal{N}_1^{l_1 - m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k - m_k} \otimes \pi^* \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)} \right) \quad \pi_* \text{ is exact} \\ & \simeq \int \pi_* (\mathcal{N}_1^{l_1 - m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k - m_k}) \otimes \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)} \quad \text{projection formula} \\ & \simeq \int \mathcal{O}(n_1 p_1)^{\lfloor \frac{l_1 - m_1}{r_1} \rfloor} \otimes \cdots \otimes \mathcal{O}(n_k p_k)^{\lfloor \frac{l_k - m_k}{r_k} \rfloor} \otimes \mathcal{F}_{\left(\frac{l_1}{r_1}, \dots, \frac{l_k}{r_k}\right)} \\ & \simeq \int \mathcal{F}_{\left(\frac{l_1}{r_1} - \lfloor \frac{l_1 - m_1}{r_1} \rfloor, \dots, \frac{l_k}{r_k} - \lfloor \frac{l_k - m_k}{r_k} \rfloor\right)} \\ & \simeq \mathcal{F}_{\left(\frac{m_1}{r_1}, \dots, \frac{m_k}{r_k}\right)}. \end{aligned}$$

6. THE PARABOLIC PULLBACK

Consider a morphism $f : Y \rightarrow X$ of smooth projective curves. We obtain a diagram

$$\begin{array}{ccc}
Y_{f^*\mathbb{D}, \vec{r}} & \xrightarrow{g} & X_{\mathbb{D}, \vec{r}} \\
\pi_Y \downarrow & & \downarrow \pi_X \\
Y & \xrightarrow{f} & X
\end{array}$$

There are associated equivalences of categories

$$\mathbf{F}_{\mathbb{D}, \vec{r}}^X : \text{Vect}(X_{\mathbb{D}, \vec{r}}) \rightarrow \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r})$$

and

$$\mathbf{F}_{\mathbb{D}, \vec{r}}^Y : \text{Vect}(Y_{\mathbb{D}, \vec{r}}) \rightarrow \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}).$$

Further there is an obvious pullback functor

$$f^* : \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}) \rightarrow \text{Vect}_{\text{par}}(f^*\mathbb{D}, \vec{r}).$$

Proposition 6.1. *We have $f^* \circ \mathbf{F}_{\mathbb{D}, \vec{r}}^X = \mathbf{F}_{f^*\mathbb{D}, \vec{r}}^Y \circ g^*$.*

Proof. This is by flat base change. \square

We will frequently apply the correspondence described in 4.2, in what follows.

Set $\vec{r} = (r_1, \dots, r_k)$, $\mathbb{D} = (n_1 p_1, \dots, n_k p_k)$ and $\vec{n} = (n_1, \dots, n_k)$. Consider an \vec{r} -divisible parabolic bundle \mathcal{F}_\bullet with parabolic structure along \mathbb{D} . Using 4.2 we have a filtration

$$F_{i,1} \supset \dots \supset F_{i,m_i} \supset F_{i,m_i+1} = 0$$

and weights

$$0 \leq \alpha_{i,1} = \frac{s_{i1}}{r_i} < \dots < \alpha_{i,m_i} = \frac{s_{im_i}}{r_i} < 1.$$

Write $n_i s_{ij} = a_{ij} r_i + e_{ij}$ with $0 \leq e_{ij} < r_i$. We also denote by \mathcal{F}_{ij} the preimage of F_{ij} in $\mathcal{F}_{(0,0,\dots,0)}$. For $x \in \frac{1}{r_i} \mathbb{Z} \cap [0, 1)$ define a subsheaf $W_{ij}^x(\mathcal{F}_\bullet)$ of $\mathcal{F}_{(0,\dots,0)}(n_i p_i)$ by

$$W_{ij}^x(\mathcal{F}_\bullet) = \begin{cases} \mathcal{F}_{(0,\dots,0)}(a_{ij} p_i) + \mathcal{F}_{i,j+1}(n_i p_i) & \text{if } x \leq \frac{e_{ij}}{r_i} \\ \mathcal{F}_{(0,\dots,0)}((a_{ij} - 1) p_i) + \mathcal{F}_{i,j+1}(n_i p_i) & \text{otherwise} \end{cases}$$

We have a subsheaf

$$\mathcal{F}_i^x = \bigcap_j W_{ij}^x(\mathcal{F}_\bullet)$$

of $\mathcal{F}_{(0,\dots,0)}(n_i p_i)$.

When $x \geq 0$, we construct subsheaves $\sqrt[n]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)}$ of

$$\mathcal{F}_{(0,\dots,0)}(n_1 p_1 + \dots + n_k p_k)$$

by setting

$$\sqrt[n]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)} = (\cap_j W_{ij}^x(\mathcal{F}_\bullet)) + \sum_{i \neq k} \mathcal{F}_k^0 = \mathcal{F}_i^x + \sum_{i \neq k} \mathcal{F}_k^0,$$

where the non-zero entry of the tuple is at the i th position. If $a_{i(j+1)} = a_{ij}$ then $e_{i,j+1} > e_{ij}$. Hence we have that $x \leq y$ implies

$$\sqrt[n]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)} \supseteq \sqrt[n]{\mathcal{F}_\bullet}_{(0,\dots,0,y,0,\dots,0)}.$$

This extends to a uniquely to a parabolic bundle

$$\sqrt[n]{\mathcal{F}_\bullet} : \left(\frac{1}{\vec{r}} \mathbb{Z} \right)^{\text{op}} \rightarrow \text{Vect}(X).$$

Setting $\frac{\vec{r}}{d} = \left(\frac{r_1}{d_1}, \dots, \frac{r_k}{d_k}\right)$ where $d_i = \gcd(r_i, n_i)$, note that this parabolic bundle is really $\frac{\vec{r}}{d}$ -divisible!

Set $\mathbb{D}_{red} = (p_1, \dots, p_k)$. We have a diagram

$$\begin{array}{ccc} X_{(\mathbb{D}_{red}, \frac{\vec{r}}{d})} & \xrightarrow{\alpha} & X_{(\mathbb{D}, \vec{r})} \\ & \searrow \pi & \swarrow \pi_n \\ & X. & \end{array}$$

There are associated equivalences

$$\mathbf{F} : \text{Vect}(X_{\mathbb{D}_{red}, \vec{r}/d}) \xleftarrow{\quad} \text{Vect}_{\text{par}}(\mathbb{D}_{red}, \vec{r}/d) : \mathbf{G}$$

and

$$\mathbf{F}_n : \text{Vect}(X_{\mathbb{D}, \vec{r}}) \xleftarrow{\quad} \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}) : \mathbf{G}_n.$$

In the remainder of this section will be devoted to proving that for a vector bundle \mathcal{F} on $X_{(\mathbb{D}, \vec{r})}$ we have

$$\sqrt[n]{\mathbf{F}_n(\mathcal{F})} \cong \mathbf{F}(\alpha^*(\mathcal{F})).$$

In order to motivate the proof and understand the definition above we compute some examples.

Example 6.2. We assume that there is only one parabolic point p with parabolic divisor np having r -divisible weights. Also set $d = \gcd(r, n)$. Consider the root line bundle \mathcal{N}^w with $0 < w < r$ on $X_{np, r}$. A calculation shows that

$$\begin{aligned} \mathbf{F}_n(\mathcal{N}^w) & : \frac{l}{r} \mapsto \mathcal{O}(np)^{\lfloor \frac{w-l}{r} \rfloor} \\ \mathbf{F}(\alpha^* \mathcal{N}^w) & : \frac{dl}{r} \mapsto \mathcal{O}(p)^{\lfloor \frac{nw-dl}{r} \rfloor}. \end{aligned}$$

Let's compute $\sqrt[n]{\mathbf{F}_n(\mathcal{N}^w)}$. Write $wn = ar + e$. The filtration of $\mathbf{F}_n(\mathcal{N}^w)_0$ is given by

$$\mathcal{F}_1 = \mathcal{O} \quad \mathcal{F}_2 = \mathcal{O}(-np)$$

and the weight of \mathcal{F}_1 is w/r . So

$$W_1^x = \begin{cases} \mathcal{O}(ap) & 0 \leq x \leq e/r \\ \mathcal{O}((a-1)p) & e/r < x < 1. \end{cases}$$

Hence

$$(\sqrt[n]{\mathbf{F}_n(\mathcal{N}^w)})_x = \begin{cases} \mathcal{O}(ap) & 0 \leq x \leq e/r \\ \mathcal{O}((a-1)p) & e/r < x < 1. \end{cases}$$

which agrees with $\mathbf{F}(\alpha^* \mathcal{N}^w)$.

Let us compute a rank two example. Consider the bundle

$$\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2}$$

with $0 < w_1 < w_2 < r$. A calculation shows that

$$\begin{aligned} \mathbf{F}_n(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2}) & : \quad \frac{l}{r} \mapsto \mathcal{O}(np)^{\lfloor \frac{w_1-l}{r} \rfloor} \oplus \mathcal{O}(np)^{\lfloor \frac{w_2-l}{r} \rfloor} \\ \mathbf{F}(\alpha^*(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2})) & : \quad \frac{dl}{r} \mapsto \mathcal{O}(p)^{\lfloor \frac{nw_1-dl}{r} \rfloor} \oplus \mathcal{O}(np)^{\lfloor \frac{nw_2-dl}{r} \rfloor} \end{aligned}$$

Let's compute $\sqrt[n]{\mathbf{F}_n(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2})}$. Write $w_j n = a_j r + e_j$. The filtration of $\mathbf{F}_n(\mathcal{N}^w)_0$ is given by

$$\begin{aligned} \mathcal{F}_1 & = \mathcal{O} \oplus \mathcal{O} \\ \mathcal{F}_2 & = \mathcal{O}(-np) \oplus \mathcal{O} \\ \mathcal{F}_3 & = \mathcal{O}(-np) \oplus \mathcal{O}(-np) \end{aligned}$$

and the weight of \mathcal{F}_j is w_j/r when $j = 1, 2$. So

$$W_1^x = \begin{cases} \mathcal{O}(a_1 p) \oplus \mathcal{O}(np) & 0 \leq x \leq e_1/r \\ \mathcal{O}((a_1 - 1)p) \oplus \mathcal{O}(np) & e_1/r < x < 1. \end{cases}$$

and

$$W_2^x = \begin{cases} \mathcal{O}(a_2 p) \oplus \mathcal{O}(a_2 p) & 0 \leq x \leq e_2/r \\ \mathcal{O}((a_2 - 1)p) \oplus \mathcal{O}((a_2 - 1)p) & e_2/r < x < 1. \end{cases}$$

Notice that $a_1 \leq a_2$ and equality implies $e_1 < e_2$. So we see that $\sqrt[n]{\mathbf{F}\alpha^*(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2})}$ agrees with $\mathbf{F}(\alpha^*\mathcal{N}^w)$.

Proposition 6.3. *Let \mathcal{F} be a vector bundle on $X_{\mathbb{D}, \vec{r}}$. Then there is a canonical inclusion*

$$\pi_* \alpha^* \mathcal{F} \subset \pi_{n*} \mathcal{F}(n_1 p_1 + \dots + n_k p_k)$$

Proof. We denote the canonical line bundles on $X_{\mathbb{D}, \vec{r}}$ by

$$\mathcal{N}_{1, \vec{n}}, \mathcal{N}_{2, \vec{n}}, \dots, \mathcal{N}_{k, \vec{n}}.$$

We have a diagram

$$\begin{array}{ccc} \alpha_* \alpha^* \mathcal{F} & \longrightarrow & \alpha_* \alpha^* (\mathcal{F} \otimes \mathcal{N}_{n_1}^{r_1} \otimes \dots \otimes \mathcal{N}_{n_k}^{r_k}) \\ \uparrow & & \uparrow \\ \mathcal{F} & \longrightarrow & \mathcal{F} \otimes \mathcal{N}_{1, \vec{n}}^{r_1} \otimes \dots \otimes \mathcal{N}_{k, \vec{n}}^{r_k} \end{array}$$

We apply $\pi_{\vec{n}, *}$ to obtain a diagram

$$\begin{array}{ccc} \pi_* \alpha^* \mathcal{F} & \xrightarrow{\lambda} & \pi_* \alpha^* (\mathcal{F} \otimes \mathcal{N}_{n_1}^{r_1} \otimes \dots \otimes \mathcal{N}_{n_k}^{r_k}) \\ \uparrow & & \uparrow \mu \\ \pi_{\vec{n}, *} \mathcal{F} & \longrightarrow & \pi_{\vec{n}, *} \mathcal{F}(n_1 p_1 + n_2 p_2 + \dots + n_k p_k). \end{array}$$

The question is now local and is easily checked. \square

Theorem 6.4. *We have*

$$\sqrt[n]{(\mathbf{F}_n \mathcal{F})_\bullet} \simeq (\mathbf{F} \alpha^* \mathcal{F})_\bullet.$$

Proof. We use 4.2. Both are then subbundles of $\mathbf{F}_n \mathcal{F}_\bullet(n_1 p_1 + \dots + n_k p_k)$ and hence the question is once again local. We may assume that there is only one parabolic point. Applying 3.6 and 5.2 we can assume $(\mathbf{F}_n \mathcal{F})_\bullet$ is of the form :

$$\frac{l}{r} \mapsto (\mathcal{O}(p)^{n(\lfloor \frac{w_1-l}{r} \rfloor)})^{\oplus \rho_1} \oplus \dots \oplus (\mathcal{O}(p)^{n(\lfloor \frac{w_k-l}{r} \rfloor)})^{\oplus \rho_k}$$

with $0 \leq w_1 < w_2 < \dots < w_k < r$. Pulling back root line bundles along the morphism

$$\alpha : X_{p,r/d} \rightarrow X_{np,r}$$

we get $\alpha^*(\mathcal{N}_n) = \mathcal{N}_1^{(n/d)}$ where $d = \gcd(r, n)$. Using 5.3 it follows that $(\mathbf{F} \alpha^* \mathcal{F})_\bullet$ is the parabolic bundle

$$\frac{l}{r} \mapsto (\mathcal{O}(p)^{\lfloor \frac{nw_1-l}{r} \rfloor})^{\oplus \rho_1} \oplus \dots \oplus (\mathcal{O}(p)^{\lfloor \frac{nw_k-l}{r} \rfloor})^{\oplus \rho_k}.$$

We need to compute $\sqrt[n]{(\mathbf{F}_n \mathcal{F})_\bullet}$. We compute the value at $l = 0$. One can deduce the general result by shifting weights. So we compute :

$$\begin{aligned} W_1^0((\mathbf{F}_n \mathcal{F})_\bullet) &= (\mathcal{O}(p)^{\lfloor \frac{nw_1}{r} \rfloor})^{\oplus \rho_1} \oplus \mathcal{O}(np)^{\oplus \rho_3} \oplus \dots \oplus \mathcal{O}(np)^{\oplus \rho_k} \\ W_2^0((\mathbf{F}_n \mathcal{F})_\bullet) &= (\mathcal{O}(p)^{\lfloor \frac{nw_2}{r} \rfloor})^{\oplus \rho_1} \oplus (\mathcal{O}(p)^{\lfloor \frac{nw_2}{r} \rfloor})^{\oplus \rho_2} \oplus \mathcal{O}(np)^{\oplus \rho_4} \dots \oplus \mathcal{O}(np)^{\oplus \rho_k} \\ &\vdots \end{aligned}$$

Taking intersection we get

$$\bigcap W_j^0 = (\mathcal{O}(p)^{\lfloor \frac{nw_1}{r} \rfloor})^{\oplus \rho_1} \oplus \dots \oplus (\mathcal{O}(p)^{\lfloor \frac{nw_k}{r} \rfloor})^{\oplus \rho_k}.$$

which is what was needed. \square

7. THE CYCLIC CASE

Given a one dimensional representation V of $\mathbb{Z}/c\mathbb{Z}$ we call the integer j , $0 \leq j \leq c-1$ the *weight* of the representation if the generator $1 + c\mathbb{Z}$ acts by multiplication by $e^{2\pi j\sqrt{-1}/c}$.

Suppose that $q : X \rightarrow Y$ is a G -cover, ramified at points p_1, \dots, p_k of Y . Suppose that the ramification index at p_i is r_i and set $\vec{r} = (r_1, \dots, r_k)$. Also, set $\mathbb{D} = (p_1, \dots, p_k)$. By combining the results 2.6, 3.3 and 5.2 we may view the cover as a tensor functor

$$\mathcal{F}_q : \text{Rep-}G \rightarrow \text{Vect}_{\text{par}}(Y, \mathbb{D}, \vec{r}).$$

If we choose preimages $q_i \in X$ of the p_i we obtain cyclic subgroups $\mathbb{Z}/r_i\mathbb{Z}$ of G that correspond to the stabilizers of q_i . We canonically identify the stabilizer with $\mathbb{Z}/r_i\mathbb{Z}$ by insisting that the stabilizer acts on the fiber of the sheaf $\mathcal{O}(-q_i)$ at q_i with weight one.

Fix an irreducible representation V of G . At each point p_i we have a weight space decomposition of

$$V = \oplus_j W_j^i$$

coming from the induced action of the stabilizers $\mathbb{Z}/r_i\mathbb{Z}$. The spaces W_j^i are representations of $\mathbb{Z}/r_i\mathbb{Z}$ and the generator of the group $\mathbb{Z}/r_i\mathbb{Z}$ acts by multiplication by $e^{2\pi j\sqrt{-1}/r_i}$. The numbers j do not depend upon the choice of preimage q_i .

Proposition 7.1. *In the terminology of 4.2, the weights of the $\mathcal{F}_q(V)_\bullet$ at p_i are j/r_i . In other words, consider tuples*

$$I = (0, \dots, 0, \underset{i\text{th}}{\frac{j}{r_i}}, 0, \dots, 0) \quad I' = (0, \dots, 0, \underset{i\text{th}}{\frac{j+1}{r_i}}, 0, \dots, 0).$$

Then

$$\mathcal{F}_q(V)_I = \mathcal{F}_q(V)_{I'}$$

$$\text{iff } W_j^i = 0.$$

Proof. By Proposition 3.3 we have a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & [X/G] & \xrightarrow{\sim} & Y_{(\mathbb{D}, \vec{r})} \\ & \searrow \pi' & \downarrow \pi & & \\ & & Y & & \end{array}$$

If \mathcal{E} is a G -equivariant bundle on X which is the pullback of some $\tilde{\mathcal{E}}$ on $[X/G]$, then $\pi_*(\tilde{\mathcal{E}}) = \pi'_*(\mathcal{E})^G$. Set $D_i = \pi^*(p_i)_{\text{red}}$. Hence

$$\pi_*(\mathcal{N}_1^{l_1} \otimes \dots \otimes \mathcal{N}_k^{l_k} \otimes \tilde{\mathcal{E}}) = \pi'_*(\mathcal{O}(l_1 D_1) \otimes \dots \otimes \mathcal{O}(l_k D_k) \otimes \mathcal{E})^G.$$

The question is now local. In formal neighbourhoods of q_i and p_i the morphism comes from a morphism of algebras of the form

$$\begin{array}{ccc} k[[t]] & \rightarrow & k[[s]] \\ t & \mapsto & s^{r_i}. \end{array}$$

The group action is by multiplication by roots of unity. Computing invariants gives the result. \square

Denote by F_m a free group on the symbols x_1, \dots, x_m . Consider the surjection $q : F_m \rightarrow \mathbb{Z}/c\mathbb{Z}$ that sends $x_i \mapsto 1$. There is an associated cover $X_q \rightarrow \mathbb{P}^1$ ramified possibly at $\{p_1, \dots, p_m\} \cup \{\infty\}$ for some $p_i \in \mathbb{P}^1 \setminus \{\infty\}$. Set $\vec{c} = (c, \dots, c, \frac{c}{\gcd\{c, m\}}) \in \mathbb{Z}^{m+1}$, $\mathbb{D} = (p_1, \dots, p_m, \infty)$, and $D = p_1 + \dots + p_m + \infty$. For the remainder of this section V_j will denote the one dimensional representation of $\mathbb{Z}/c\mathbb{Z}$ where $1 + c\mathbb{Z}$ acts by multiplication by $e^{2\pi j\sqrt{-1}/c}$. Set

$$\mathcal{F}_{X_q}(V_j)_{(0, \dots, 0)} =: \mathcal{O}(s_j),$$

where s_j is some integer. Also, let w_j denote the rational number in $[0, 1)$ which differs from $-\frac{mj}{c}$ by an integer.

The purpose of this section is to describe the functor \mathcal{F}_{X_q} . To this end, in the above proposition take $X = X_q$, $Y = \mathbb{P}^1$, $G = \mathbb{Z}/c\mathbb{Z}$, $k = m + 1$, $D_j = p_j$ for $1 \leq j \leq m$, $D_{m+1} = \infty$, and $\mathcal{F}_q(V_j) = \mathcal{F}_{X_q}(V_j)_\bullet$. This gives

Corollary 7.2. *Let $t = \frac{a}{\gcd(m, c)}$ and suppose $0 \leq t \leq w_j$. Then*

$$\mathcal{F}_{X_q}(V_j)_{(0, \dots, 0, t)} = \mathcal{O}(s_j),$$

and

$$\mathcal{F}_{X_q}(V_j)_{(0, \dots, 0, w_j + \frac{\gcd(m, c)}{c})} = \mathcal{O}(s_j)(-\infty).$$

Moreover, if the non-zero entry of the tuple is at the i th position for $1 \leq i \leq m$,

$$\mathcal{F}_{X_q}(V_j)_{(0, \dots, 0, \frac{i+1}{c}, 0, \dots, 0)} = \mathcal{O}(s_j)(-p_i),$$

but

$$\mathcal{F}_{X_q}(V_j)_{(0, \dots, 0, \frac{i}{c}, 0, \dots, 0)} = \mathcal{O}(s_j).$$

Let δ_{ij} denote the Kronecker delta function.

Lemma 7.3. *If $1 \leq w_1 + w_j$ then*

$$(\mathcal{F}_{X_q}(V_1)_\bullet \otimes \mathcal{F}_{X_q}(V_j)_\bullet)_{(0, \dots, 0)} = \mathcal{O}(s_1 + s_j + 1 + m\delta_{c-1, j}).$$

Otherwise,

$$(\mathcal{F}_{X_q}(V_1)_\bullet \otimes \mathcal{F}_{X_q}(V_j)_\bullet)_{(0, \dots, 0)} = \mathcal{O}(s_1 + s_j + m\delta_{c-1, j}).$$

Proof. Consider $t \in \frac{\gcd(m, c)}{c}\mathbb{Z}$ and set

$$\vec{t} = (0, \dots, 0, t).$$

Write $t = n + f$ where $f \in [0, 1)$. We compute

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}).$$

The possibilities are

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \begin{cases} \mathcal{O}(s_1 + s_j + 1) \\ \mathcal{O}(s_1 + s_j) \\ \mathcal{O}(s_1 + s_j - 1) \\ \mathcal{O}(s_1 + s_j - 2) \end{cases}$$

We are interested in when the first possibility occurs as the second occurs at $t = 0$ so when we take the sheaf generated by all possible tensor products the value will be at least this sheaf.

Suppose that $1 \leq w_1 + w_j$. Now take $t = 1 - w_j$. Then

$$\mathcal{F}_{X_q}(V_j)_{-\vec{t}} = \mathcal{O}(s_j + 1).$$

and

$$\mathcal{F}_{X_q}(V_1)_{\vec{t}} = \mathcal{O}(s_1).$$

Conversely, suppose that

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j + 1).$$

We have either

$$w_1 - 1 \leq w_j - 1 < w_1 \leq w_j$$

or

$$w_j - 1 \leq w_1 - 1 < w_j \leq w_1.$$

We conclude that $-f \leq w_j - 1$ and $f \leq w_1$ or we must have $-f \leq w_1 - 1$ and $f \leq w_j$.

We conclude that there is a t for which

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j + 1)$$

if and only if $w_1 + w_j \geq 1$.

Now we turn our attention to the other parabolic points. We preserve the notation above except we set

$$\vec{t} = (0, \dots, 0, t, 0, \dots, 0)$$

and now $t \in \frac{1}{c}\mathbb{Z}$. We have a chain of inequalities

$$\frac{1}{c} - 1 \leq \frac{j}{c} - 1 < \frac{1}{c} \leq \frac{j}{c}.$$

Suppose firstly that $j < c - 1$. Then if $-f \leq \frac{j}{c} - 1$ we have $f \geq 1 - \frac{j}{c} > \frac{1}{c}$. If $-f = \frac{1}{c} - 1$ then $f > \frac{j}{c}$. It follows that

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j).$$

When $j < c - 1$, the result follows by putting this together.

Now fix $j = c - 1$. Set

$$\vec{u} = (u_1, \dots, u_m, u_{m+1})$$

where $u_i \in \frac{1}{c}\mathbb{Z}$ for $1 \leq i \leq m$ and $u_{m+1} \in \frac{\gcd(m, c)}{c}$, and write $u_i = n_i + f_i$ where $f_i \in [0, 1)$.

Here, computing

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} \otimes \mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}}$$

the possibilities are

$$\mathcal{O}(s_1 + s_{c-1} + g(\vec{u}))$$

where $g(\vec{u})$ ranges over all integers from -2 to $m+1$: Indeed, as before, the parabolic point at infinity gives at most a contribution of $+1$ to $g(\vec{u})$ and at least -2 , while each finite parabolic point contributes either 0 or $+1$.

At the same time,

$$(1) \quad \mathcal{F}_{X_q}(V_1)_{(\frac{1}{c}, \dots, \frac{1}{c}, 0)} \otimes \mathcal{F}_{X_q}(V_{c-1})_{(-\frac{1}{c}, \dots, -\frac{1}{c}, 0)} = \mathcal{O}(s_1 + s_{c-1} + m).$$

This means that

$$(\mathcal{F}_{X_q}(V_1)_{\bullet} \otimes \mathcal{F}_{X_q}(V_{c-1})_{\bullet})_{(0, \dots, 0)} \supseteq \mathcal{O}(s_1 + s_{c-1} + m)$$

from the definition of parabolic tensor product.

Hence, we need only determine when $g(\vec{u}) = m + 1$.

Suppose that $1 \leq w_1 + w_{c-1}$. Then if $\vec{u} = (\frac{1}{c}, \dots, \frac{1}{c}, 1 - w_{c-1})$,

$$\mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}} = \mathcal{O}(s_{c-1} + m + 1)$$

and

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} = \mathcal{O}(s_1).$$

Conversely, suppose that there exists some \vec{u} such that

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} \otimes \mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}} = \mathcal{O}(s_1 + s_{c-1} + m + 1).$$

This case only occurs when either $-f_{m+1} \leq w_{c-1} - 1$ and $f_{m+1} \leq w_1$ or $-f_{m+1} \leq w_1 - 1$ and $f_{m+1} \leq w_{c-1}$ by the same argument as before. Hence, necessarily, $w_1 + w_{c-1} \geq 1$. \square

Remark 7.4. $\mathcal{F}_{X_q}(V_j)_{\bullet}$ is the j th parabolic tensor power of $\mathcal{F}_{X_q}(V_1)_{\bullet}$: Indeed, since \mathcal{F}_{X_q} is a tensor functor, we must have $\mathcal{F}_{X_q}(V_1)_{\bullet}^{\otimes c} = \mathcal{F}_{X_q}(V_1^{\otimes c})_{\bullet} = \mathcal{F}_{X_q}(V_0)_{\bullet}$, the trivial parabolic bundle. Similarly, $\mathcal{F}_{X_q}(V_1)_{\bullet}^{\otimes l} = \mathcal{F}_{X_q}(V_j)_{\bullet}$ whenever $l \equiv j$ modulo c .

In order to determine $\mathcal{F}_{X_q}(V_j)_{\bullet}$ it thus suffices to compute s_1 .

For each j with $1 \leq j \leq c-1$, set

$$\kappa_{m,c}^{(j)} = \begin{cases} 1 & \text{when } w_1 + w_j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\kappa_{m,c} = \sum_{j=1}^{c-1} \kappa_{m,c}^{(j)} = |\{j : 1 \leq j \leq c-1, w_1 + w_j \geq 1\}|.$$

Theorem 7.5. *With notation as above,*

$$s_1 = -\frac{m + \kappa_{m,c}}{c}$$

Proof. Applying Lemma 7.3 iteratively, along with Remark 7.4, one finds that

$$\mathcal{O}(s_{c-1}) = \mathcal{O}((c-1)s_1 + \kappa_{m,c} - \kappa_{m,c}^{(c-1)}).$$

Repeat the calculation once more (in the special case that $j = c-1$) to obtain

$$\mathcal{O}(s_c) = \mathcal{O}(cs_1 + \kappa_{m,c} + m).$$

The result now follows. \square

The proof of 7.5 yields the

Corollary 7.6. *For $1 \leq j \leq c-1$, the s_j of 7.2 are given in terms of s_1 by*

$$s_j = js_1 + \sum_{i=1}^{j-1} \kappa_{m,c}^{(i)} = -j \left(\frac{m + \kappa_{m,c}}{c} \right) + \sum_{i=1}^{j-1} \kappa_{m,c}^{(i)}.$$

Corollary 7.7. *We have $s_0 = 0$ and $s_j \leq -1$ for $j > 0$.*

Proof. The assertion for s_0 is clear. The numbers are necessarily integers. We have, by definition $s_1 < 0$ and hence $s_1 \leq -1$. The result now follows. \square

By the above computation, $\kappa_{m,c}$ is necessarily congruent to $-m$ modulo c . This fact may be shown independently:

Lemma 7.8.

$$\kappa_{m,c} \equiv -m \text{ modulo } c.$$

Proof. When $m \equiv 0$ modulo c , it follows that $w_j = 0$ for all $1 \leq j \leq c-1$, and hence $\kappa_{m,c} = 0$.

Suppose now that $m \equiv -v$ modulo c , for some $0 < v < c$. Then $w_1 = \frac{v}{c}$, while for j with $1 \leq j \leq c-1$,

$$w_j = \begin{cases} \frac{vj}{c} & 0 < vj < c \\ \vdots & \vdots \\ \frac{vj-tc}{c} & tc \leq vj < (t+1)c \\ \vdots & \vdots \\ \frac{vj-(v-1)c}{c} & (v-1)c \leq vj < vc. \end{cases}$$

For t with $0 \leq t \leq c-1$, then $tc \leq vj < (t+1)c$ implies $0 \leq vj - tc < c$. Now let j_t be the largest integer value of j satisfying this inequality. Then $v(j_t + 1) - tc \geq c$, so that

$$w_1 + w_{j_t} = \frac{v(1 + j_t) - tc}{c} \geq 1.$$

At the same time, for any integer j satisfying the inequality which also has $j < j_t$, then $j + 1 \leq j_t$ and necessarily

$$w_1 + w_j \leq \frac{vj_t - tc}{c} < 1.$$

So among the integers j such that $tc \leq vj < (t+1)c$, there is exactly one with $w_1 + w_j \geq 1$. There are exactly v such inequalities, so $\kappa_{m,c} = v$. \square

8. REDUCTION TO THE CYCLIC CASE

Suppose that $X_q \rightarrow \mathbb{P}^1$ is a Galois covering with $\text{Deck}(X_q/\mathbb{P}^1) = G$ ramified at $0, 1$ and ∞ . Let $q : F_2 \twoheadrightarrow G$ denote the corresponding surjection and $\mathbb{T} = (0, 1, \infty)$. Then as before, by 2.6, 3.3 and 5.2 the cover may be viewed as a functor

$$F_{X_q} : \text{Rep-}G \rightarrow \text{Vect}_{\text{par}}(\mathbb{P}^1, \mathbb{T}).$$

Our goal in this section is to produce a bound on the u_j for which

$$F_{X_q}(V)_{(0, \dots, 0)} = \mathcal{O}(u_1) \oplus \dots \oplus \mathcal{O}(u_k)$$

for a fixed $V \in \text{Ob}(\text{Rep-}G)$.

The idea is to reduce to the cyclic case by delooping the ramification at 0 as follows: Suppose that the ramification index at 0 is m - i.e. under the mapping q , the image of the generator of F_2 corresponding to a loop about 0 in $\pi_1(\mathbb{P}^1)$ has order m in G . Form the base change

$$\begin{array}{ccc} X_q \times_{\mathbb{P}^1} \mathbb{P}^1 & \xrightarrow{\quad} & X_q \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{z \mapsto z^m} & \mathbb{P}^1 \end{array}$$

and denote the desingularization of $X_q \times_{\mathbb{P}^1} \mathbb{P}^1$ by Y . Now $Y \rightarrow \mathbb{P}^1$ ramifies at ∞ and the m th roots of unity, μ_m . Hence Y corresponds to a homomorphism $h : F_m \rightarrow G$ which factors through F_2 by mapping the generators of F_m corresponding to each root of unity to the generator σ_1 of F_2 corresponding to 1.

Then the image of h is generated by $q(\sigma_1)$, which is a cyclic subgroup of G , say $\mathbb{Z}/c\mathbb{Z}$.

We have a decomposition $Y = \coprod_{\tau \in G/\text{Im}(h)} Y_\tau$ where the Y_τ are all cyclic covers.

Using the argument at the start of §7, we obtain a tensor functor

$$F_Y : \text{Rep-}G \rightarrow \text{Vect}_{\text{par}}(\mathbb{P}^1, (\mu_m, \infty)).$$

Lemma 8.1. *The functor F_Y factors as*

$$\begin{array}{ccc} \text{Rep-}G & \xrightarrow{F_Y} & \text{Vect}_{\text{par}}(\mathbb{P}^1, (\mu_m, \infty)) \\ \downarrow & \nearrow F_{Y_e} & \\ \text{Rep-}\mathbb{Z}/c\mathbb{Z} & & \end{array}$$

Proof. The functors are computed by taking invariants as in the proof of 7.1. The result now follows from the disjoint union above. \square

We need :

Proposition 8.2. *If $\mathbb{D} = (p_1, \dots, p_k)$ with $\vec{r} = (r_1, \dots, r_k)$, and $\mathbb{D}' = (p_0, p_1, \dots, p_k)$ with $\vec{r}' = (1, r_1, \dots, r_k)$, then there exist natural equivalences of tensor categories*

$$\mathbf{F}' : \text{Vect}_{\text{par}}(\mathbb{D}', \vec{r}') \xrightarrow{\sim} \text{Vect}_{\text{par}}(\mathbb{D}, \vec{r}) : \mathbf{G}'.$$

Proof. The root stacks $X_{\mathbb{D}, \vec{r}}$ and $X_{\mathbb{D}', \vec{r}'}$ are isomorphic. Now invoke Theorem 5.2. \square

Remark 8.3. Let ζ_m denote a primitive m th root of unity. Then in the notation of 8.2 set $\mathbb{D} = (\zeta_m, \zeta_m^2, \dots, \zeta_m^{m-1}, 1, \infty)$ and $\vec{r} = (c, \dots, c, \frac{c}{\gcd(m, c)})$. Also take $p_0 = 0$. By 3.5 and 6.4 we have that $f_{\text{par}}^*(F_{X_q}) = \mathbf{G}' F_Y$.

Since \mathbf{G}' is an equivalence of tensor categories, the constants computed in section 7 pertaining to F_Y are the same as those relating to $\mathbf{G}' F_Y$.

We denote by $\kappa_{m, c}$ and $\kappa_{m, c}^{(i)}$ the numbers defined before Theorem 7.5 for the cover $Y_e \rightarrow \mathbb{P}^1$. We will also make use of the notation set up after 6.1. In particular, let a_1 denote the minimum among the a_{i1} . Further denote by a_0 and a_∞ a_{i1} for the index i corresponding to the points 0 and ∞ respectively.

The representation V viewed as a representation of $\mathbb{Z}/c\mathbb{Z}$ decomposes into weight spaces :

$$V = V_{j_1} \oplus \dots \oplus V_{j_k}.$$

We have

$$F_{Y_e}(V)_{(0, \dots, 0)} = \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(t_k)$$

where the t_i are computed in 7.5 and 7.6. We may reindex so that

$$t_1 \leq t_2 \leq \dots \leq t_k \leq 0.$$

The last inequality is by 7.7.

Theorem 8.4. *With the above notation, consider*

$$F_{X_q}(V)_{(0, \dots, 0)} = \mathcal{O}(u_1) \oplus \dots \oplus \mathcal{O}(u_k).$$

We reindex so that

$$u_1 \leq u_2 \leq \dots \leq u_k.$$

Then the u_j are bounded above as follows:

$$u_j \leq \frac{t_j}{m} - \frac{a_0}{m} - \frac{a_\infty}{m}.$$

(Hence the u_j are negative, by 7.7.)

Proof. We have

$$f^*(F_{X_q}(V)_{(0, \dots, 0)}) = \mathcal{O}(mu_1) \oplus \dots \oplus \mathcal{O}(mu_k).$$

With ζ_m denoting a primitive m th root of unity as above, the curve Y ramifies over

$$p_1 = \zeta_m, \dots, p_m = \zeta_m^m = 1, p_{m+1} = \infty.$$

By 8.3 the parabolic pullback of $F_{X_q}(V)_\bullet$ also has 1-divisibility at $p_0 := 0$.

Now by the definition of parabolic pullback, $f_{\text{par}}^* F_{X_q}(V)_{(0, \dots, 0)}$ contains the intersection $\cap_j W_{ij}^0$. Hence

$$f_{\text{par}}^* F_{X_q}(V)_{(0, \dots, 0)} \supseteq (f^*(F_{X_q}(V)_{(0, \dots, 0)})(a_{i1}))$$

as $a_{i1} \leq a_{ij}$. Notice that

$$a_{11} = \dots a_{m1} = a_1.$$

Hence

$$\begin{aligned} & \mathcal{O}(mu_1) \oplus \dots \oplus \mathcal{O}(mu_k)(a_0.0 + a_\infty.\infty + \sum a_1 p_i) \\ \simeq & \mathcal{O}(mu_1 + a_0 + ma_1 + a_\infty) \oplus \dots \oplus \mathcal{O}(mu_k + a_0 + ma_1 + a_\infty) \\ \subseteq & f_{\text{par}}^* F_{X_q}(V)_{(0, \dots, 0)} \\ = & \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(t_k). \end{aligned}$$

The result now follows from 8.5 below and observing that $a_1 = 0$. \square

Lemma 8.5. *If $\mathcal{O}(s_1) \oplus \dots \oplus \mathcal{O}(s_u) \subseteq \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(t_u)$, there exists $\sigma \in S_u$ such that $s_{\sigma(j)} \leq t_j$ for all j with $1 \leq j \leq u$.*

Proof. When $u = 1$, this is well-known. Proceeding by induction, suppose that the assertion is known to be valid for all $u \leq N - 1$. Then consider an injection

$$\phi : \mathcal{O}(s_1) \oplus \dots \oplus \mathcal{O}(s_N) \hookrightarrow \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(t_N)$$

where the s_j and t_j may be taken to be ordered - i.e. $s_1 \leq \dots \leq s_N$ and $t_1 \leq \dots \leq t_N$. Necessarily, $s_N \leq t_L$ for some L , but if $s_N \leq t_1$ we are done. Suppose then that there exists some i such that $t_{i-1} < s_N \leq t_i$. For j with $i \leq j \leq N$, consider the mapping

$$\phi_j : \mathcal{O}(s_1) \oplus \dots \oplus \mathcal{O}(s_{N-1}) \rightarrow \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(\hat{t_j}) \oplus \dots \oplus \mathcal{O}(t_N)$$

induced from ϕ . Should there exist j for which ϕ_j is injective, we are done by the inductive hypothesis. Suppose to the contrary that for every j , ϕ_j is not injective. Then we can show this implies the original ϕ could not have been injective: Indeed, $s_N > t_{i-1}$ implies that under ϕ , the restricted morphism $\mathcal{O}(s_N) \rightarrow \mathcal{O}(t_1) \oplus \dots \oplus \mathcal{O}(t_{i-1})$ is zero.

Passing to the generic point of the curve the morphism ϕ is given by an $N \times N$ matrix. The last row of this matrix begins with $i - 1$ zero entries. Computing the determinant of ϕ by cofactor expansion along this row, we find

$$\det \phi = 0 + \det \phi_i \cdot \gamma_i + \dots + \det \phi_N \cdot \gamma_N$$

for some constants γ_j . Hence the morphism at the generic point is not injective. This is a contradiction as pullback to the generic point is flat. \square

Example 8.6. Denote by Q_8 the quaternion group of order 8. It has a two dimensional representation given in terms of matrices by

$$\begin{aligned} i & \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \\ j & \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ k & \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \end{aligned}$$

Consider the quotient $F_2 \twoheadrightarrow Q_8$ with $x_0 \mapsto j$, $x_1 \mapsto i$. As x_1 has a weight 3 eigenspace we have $t_1 = -3$. Both a_1 and a_∞ are 1. Hence $u_1 \leq -2$.

It follows from the lower bound in [3, theorem 5.12] that u_1 must be -2.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON ONTARIO N6A 5B7

E-mail address: `adhill13@uwo.ca`

E-mail address: `sjoyner@uwo.ca`